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LAGRANGE DUAL PROBLEMS WITH LINEAR CONSTRAINTS ON THE MULTIPLIE--ETC(U)
JUN 78 C E BLAIR, R G JEROSLOW N00014-75-C-0621

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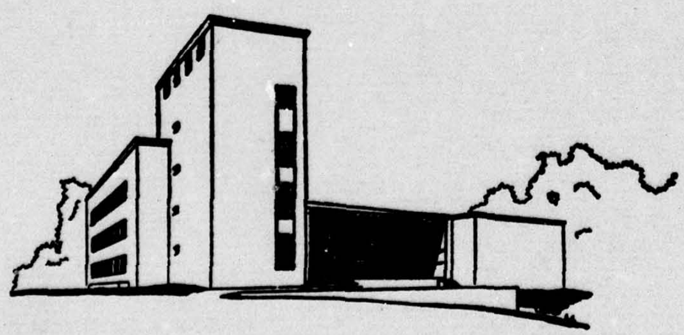

LEVEL II

(9) Management Sciences Research Report No. 419

(6) LAGRANGE DUAL PROBLEMS WITH LINEAR CONSTRAINTS ON THE MULTIPLIERS.

by

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(11) June [redacted] 78 (12) 28p.

(14) MSRR-419

(15) N00014-75-C-0621, NSF-GP-37510

This report was prepared as part of the activities of the Management Sciences Research Group, Carnegie-Mellon University, under Grant #GP 37510 X1 of the National Science Foundation and Contract N00014-75-C-0621 NR047-048 with the U.S. Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the U.S. Government.

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Abstract

We consider Lagrange dual problems in which the vector of Lagrange multipliers, in addition to the usual nonnegativity conditions, satisfies certain linear constraints. Of course, the maximum dual value, for such restricted classes of multipliers, need not equal the primal value, even if various "constraint qualifications" are appended.

We obtain expressions for this maximum "constrained dual" value, in terms of perturbation functions for the primal convex program, and value functions associated with the linear constraints. At least some information on the primal perturbation functions appears to be necessary in order to put upper or lower bounds on the "constrained dual" value. For the case that the constraints are, simply, that a positive weighted sum of the multipliers shall not exceed some given bound (plus, of course, the usual nonnegativities), only two values of an associated one-dimensional perturbation function is needed to obtain such upper and lower bounds.

Key Words:

1) Lagrangeans

2) Duality

3) Convexity

Lagrange Dual Programs with Linear Constraints on the Multipliers

by

C. E. Blair and R. G. Jeroslow

Dedicated to Dick Duffin

Let S be a nonempty convex subset of a vector space and $f, g_1, \ldots, g_k : S \to R$ be convex (not necessarily continuous) functions. An ordinary constrained convex optimization program (see e.g., [7, Section 28]) is

$$(P) \qquad \begin{array}{l} \text{minimize } f(x) \\ \text{subject to } g_i(x) \leq 0 \text{ for } 1 \leq i \leq k \\ \text{and } x \in S. \end{array}$$

Throughout the paper, we assume that (P) is consistent.

For any $\lambda_1, \ldots, \lambda_k \geq 0$ the Lagrange dual problem is

$$(D_\lambda) \qquad \begin{array}{l} \text{minimize } f(x) + \sum_{i=1}^{k} \lambda_i g_i(x) \\ \text{subject to } x \in S. \end{array}$$

The value $v(D_\lambda)$ of (D_λ) clearly does not exceed the value $v(P)$ of (P) and it may be $-\infty$. One uses the unconstrained optimization problem (D_λ) to give information about the constrained one (P). If $v(D) = \sup_{\lambda \geq 0} v(D_\lambda)$, then $v(D) \leq v(P)$.

In this paper, we study (D_λ) for those $\lambda \geq 0$ satisfying certain conditions, for example:

(a) $\sum_{1}^{k} \lambda_i \leq M$;

or

(b) $\lambda_i \leq M$, $1 \leq i \leq k$;

or, more generally,

(c) $A\lambda \leq b$, where $b \in R^m$ and $A \in R^{m \times k}$.

In other words if T is a finite system of linear constraints on the λ_i, we are interested in $v(D_T) = \sup \{v(D_\lambda) \mid \lambda \text{ satisfies } T\}$.

We establish relations between $v(D_T)$ and certain relaxations of (P). In the spirit of Duffin [4], Blair [1], Charnes, Cooper, and Kortanek [3], Duffin and Jeroslow [5], and Duffin and Karlovitz [6], we will use theorems about systems of linear inequalities systematically.

We begin with the situation in which $\{\lambda \mid A\lambda \leq b, \lambda \geq 0\}$ is bounded (i.e., a polytope). This case includes (1) and (2) above; and has simpler conclusions than the case of arbitrary A, which we treat later.

Section 1: $\{\lambda \mid A\lambda \leq b, \lambda \geq 0\}$ is bounded

Lemma 1: Let $P = \{\lambda \mid A\lambda \leq b, \lambda \geq 0\}$ be nonempty and bounded (i.e., a polytope).

There is no $\lambda \in P$ such that

$$f(x) + \sum_1^k \lambda_i g_i(x) \geq L \tag{1}$$

for every $x \in S$, if and only if, there is a single point $\underline{x} \in S$ such that the finite system of linear inequalities in unknowns $\lambda_1, \ldots, \lambda_k$

$$(S_{\underline{x}}) \qquad \begin{aligned} A\lambda &\leq b \\ \sum_{i=1}^k g_i(\underline{x})\lambda_i &\geq L - f(\underline{x}) \\ \lambda &\geq 0 \end{aligned}$$

alone has no solution.

<u>Proof</u>: The "if" part is immediate. We now prove the "only if" part.

Let $x \in S$ and denote, by $T_x \subset R^k$, the set of solutions to S_x. T_x is compact for every x, since it is closed by virtue of being a polyhedron, and bounded as it is a subset of P. Now $\bigcap_{x \in S} T_x = \emptyset$ if (1) fails. Therefore, there is a finite $F \subseteq S$ such that $\bigcap_{x \in F} T_x$ is empty, by standard compactness results. Then the system of linear inequalities

$$A\lambda \leq b$$

$$(2) \qquad \sum_{i=1}^{k} g_i(x)\lambda_i \geq L - r(x), \text{ for all } x \in F$$

$$\lambda \geq 0$$

has no solution.

By the Kuhn-Fourier Theorem [8] there are nonnegative $w \in R^m$, $u_x \in R$ such that $w^tA - \sum_{x \in F} u_x(g_1(x),\ldots,g_k(x)) \geq \vec{0}$ and $w^tb + \sum_{x \in F} u_x(f(x) - L) < 0$.

Since P is nonempty $u_x > 0$ for at least one $x \in F$. Let $N = \sum_{x \in F} u_x > 0$, and $\underline{x} = \frac{1}{N} \sum_{x \in F} u_x x$. Since $F \subseteq S$, clearly $\underline{x} \in S$. By convexity, $f(\underline{x}) \leq \frac{1}{N} \sum_{x \in F} u_x f(x)$ and $g_i(\underline{x}) \leq \frac{1}{N} \sum_{x \in F} u_x g_i(x)$. Therefore $w^tA - N(g_1(\underline{x}),\ldots,g_n(\underline{x})) \geq \vec{0}$ and $w^tb + N(f(\underline{x}) - L)) < 0$, i.e., the system $(S_{\underline{x}})$ has no solutions.

Q.E.D.

<u>Theorem 1</u>: Let P be as in the lemma. Let V: $R^k \to R$ be defined by:

$$(3) \qquad V(u) = \max \{u\lambda \mid A\lambda \leq b, \lambda \geq 0\}.$$

Let p: $R^k \to R$ be defined by:

$$(4) \qquad p(u) = \inf \{f(x) \mid g_i(x) \leq u_i \text{ for } 1 \leq i \leq k \text{ and } x \in S\},$$

where we set $p(u) = +\infty$ if there is no $x \in S$ satisfying $g_i(x) \leq u_i$ for $1 \leq i \leq k$, and $r + \infty = +\infty$ for all reals $r \in R$. Also, define

(5) $$L = \inf_{u \in R^k} \{V(u) + p(u)\}.$$

If $L = -\infty$, then $v(D_\lambda) = -\infty$ for all λ satisfying $\lambda \geq 0$ and $A\lambda \leq b$. In fact, if $L \in R \cup \{-\infty\}$, then $v(D_\lambda) \leq L$ for all λ with $\lambda \geq 0$, $A\lambda \leq b$.

Moreover, there exists $\underline{\lambda} \geq 0$ with $A\underline{\lambda} \leq b$ and $v(D_{\underline{\lambda}}) = L$.

<u>Proof</u>: First, suppose that $L \in R$.

Let $\lambda \in P$. We show that $v(D_\lambda) \leq L$. In fact, if $v(D_\lambda) > L$, then for some $\epsilon > 0$,

(6) $$f(x) + \sum_{i=1}^{k} \lambda_i g_i(x) \geq L + \epsilon \quad \text{for all } x \in S.$$

Let $u \in R^k$ be arbitrary. If $p(u) \neq +\infty$, then $p(u) \in R$, since $p(u) = -\infty$ is ruled out by $L \in R$ (note that $V(u) \in R$ for all $u \in R^k$, by inspection of (3)). For any $x \in X(u) = \{x \in S \mid g_i(x) \leq u_i \text{ for } 1 \leq i \leq k\} \neq \emptyset$, (6) gives

(6)′ $$f(x) + \sum_{i=1}^{k} \lambda_i u_i \geq L + \epsilon$$

as all $\lambda_i \geq 0$. From (6)′, it follows at once, by taking the infimum over $x \in X(u)$, that

(7) $$p(u) + \sum_{i=1}^{k} \lambda_i u_i \geq L + \epsilon.$$

Then invoking the definition (3) of $V(u)$, we have

(8) $$p(u) + V(u) \geq L + \epsilon.$$

Also, if $p(u) = +\infty$, (8) is true. Therefore, (8) holds for all $u \in R^k$, contradicting the definition (5) of L.

Continuing with our assumption $L \in R$, note that, for any $x \in S$,

$$(9) \qquad L - f(x) \leq L - p(g_1(x),\ldots,g_k(x)) \leq V(g_1(x),\ldots,g_k(x)),$$

using $u = (g_1(x),\ldots,g_k(x))$ in the definition (5) of L. From (9), (S_x) has a solution. By Lemma 1, a $\underline{\lambda}$, as described in the Theorem, exists.

Finally, suppose that $L = -\infty$. Then we repeat the discussion, in equations (6)-(8), with any finite value L' replacing $L + \epsilon$ on the r.h.s. in (6). Again, we obtain a contradiction. This proves that $v(D_\lambda) = -\infty$ if $\lambda \geq 0$ and $A\lambda \leq b$.

Q.E.D.

The function p above is the usual perturbation function [7]. V is the criterion "value function" associated with a linear program, and as is well known, it can be written as the maximum of finitely many linear affine functions, each of which corresponds to an extreme point of the feasible region.

Theorem 1 tells us how good a value we can hope for in the program (D_λ) given that the λ must lie in P. Consider a family of polytopes $P_\alpha = \{\lambda \mid A\lambda \leq \alpha b, \lambda \geq 0\}$ for $\alpha > 0$, $b \in R^m$, $b \geq \vec{0}$. As α increases, P_α becomes larger. Given L we can ask what the smallest α is such that there exist $\lambda \in P_\alpha$ with $v(D_\lambda) \geq L$. This is a "converse" of the question answered by Theorem 1.

<u>Theorem 2</u>: Let P_α be polytopes as above. Define a "generalized perturbation function" G: $R^m \rightarrow R$, depending on A, by

$$(10) \qquad G(w) = \inf \{f(x) \mid \text{subject to } x \in S \text{ and } (g_1(x),\ldots,g_k(x)) \leq w^t A\}.$$

Define a "modulus" for $L \in R$, by:

$$(11) \qquad h_b(A,L) = \sup_{w \geq 0} \frac{L - G(w)}{w^t b},$$

where we conventionally assign $\frac{0}{0} = 0$, $-(-\infty) = +\infty$, $L + \infty = +\infty$, $\frac{\alpha}{0} = -\infty$ for $\alpha < 0$, $\frac{\alpha}{0} = +\infty$ for $\alpha > 0$, $L - \infty = -\infty$, and we set $G(w) = +\infty$ if there exists no $x \in S$ satisfying $g_i(x) \leq \sum_{j=1}^{m} w_j a_{ji}$ for $i \leq 1 \leq k$.

Suppose that $\alpha \geq 0$, $\alpha \in R$, $b \geq 0$.

Then there exist $\lambda \in P_\alpha$ such that $v(D_\lambda) \geq L$ if and only if $\alpha \geq h_b(A,L)$. (In particular, if $h_b(A,L) = +\infty$, then L is not obtainable for any α; and consequently also, $v(D) \leq L$ if $b > 0$.)

<u>Proof</u>: First we establish the "only if" part.

Suppose $f(x) + \Sigma \lambda_i g_i(x) \geq L$ for all $x \in S$, i.e., $v(D_\lambda) \geq L$, with $\lambda \in P_\alpha$. Let $w \geq 0$. Choose $\epsilon > 0$ and let x_o be such that, assuming $G(w)$ finite, $(g_1(x_o),\ldots,g_n(x_o)) \leq w^t A$ and $f(x_o) \leq G(w) + \epsilon$. Since $\lambda \in P_\alpha$, $L - G(w) - \epsilon \leq L - f(x_o) \leq \Sigma \lambda_i g_i(x_o) \leq w^t A\lambda \leq \alpha(w^t b)$. Since ϵ was arbitrary, $L - G(w) \leq \alpha(w^t b)$. The same result holds if $G(w) = +\infty$; and the same reasoning shows that $G(w) = -\infty$ cannot occur (as $v(D_\lambda) = -\infty$ for all $\lambda \geq 0$).

If $w^t b \neq 0$, then $w^t b > 0$ (as $w \geq 0$ and $b \geq 0$), so $(L - G(w))/(w^t b) \leq \alpha$. If $w^t b = 0$, we have $L \leq G(w)$, and so by our conventions $(L - G(w))/(w^t b) \leq \alpha$ as $\alpha \geq 0$. Thus $h_b(A,L) \leq \alpha$, as $w \geq 0$ was arbitrary.

We now establish the "if" part. Let $\alpha \geq h_b(A,L)$. We show the system (S_x), with b replaced with αb, has solutions for all $x \in S$ and apply Lemma 1.

Let $x \in S$. By the Kuhn-Fourier Theorem, and the fact that P_α is nonempty, (S_x) has no solutions only if there is a $w \geq 0$ such that $w^t A \geq (g_1(x),\ldots,g_k(x))$ and $\alpha w^t b < L - f(x)$. Since $L - f(x) \leq L - G(w)$, we obtain $\alpha w^t b < L - G(w)$. If $w^t b > 0$, we have $\alpha < (L - G(w))/w^t b$, which contradicts $\alpha \geq h_b(A,L)$. If $w^t b = 0$, we have $G(w) < L$, hence

$(L - G(w))/w^t b = +\infty$ by our conventions. Therefore, $h_b(A,L) = +\infty$, again contradicting $\alpha \geq h_b(A,L)$.

Q.E.D.

For the case where some components of b are negative there is the result:

<u>Theorem 3</u>: Let $c \in R^m$, $c \geq 0$, $P^*_\alpha = \{\lambda \mid \lambda \geq 0, A\lambda \leq b + \alpha c\}$, $\alpha \geq 0$, $\alpha \in R$, and suppose that $P_\alpha \neq \emptyset$, and P_0 is nonempty and bounded.

Then there exist $\lambda \in P^*_\alpha$ such that $v(D_\lambda) \geq L$ iff

$$\alpha \geq \sup_{w \geq 0} \frac{L - G(w) - wb}{wc}.$$

The proof is essentially the same as that for Theorem 2, which is a special case.

Theorem 2 for $\alpha = 1$ can be rephrased in terms of conjugate functions, if we restrict G to the domain $\{w \mid w \geq 0\}$, and $P = \{\lambda \geq 0 \mid A\lambda \leq b\}$ is nonempty and bounded. Indeed, by Theorem 2, there exists $\lambda \in P$ with $v(D_\lambda) \geq L$ if and only if

(12a) $$1 \geq (L - G(w))/w^t b \quad \text{for all } w \geq 0,$$

or, equivalently,

(12b) $$w^t b \geq L - G(w) \quad \text{for all } w \geq 0, \text{ with } w^t b > 0;$$

and

(12c) $$L - G(w) \leq 0 \quad \text{for all } w \geq 0, \text{ with } w^t b = 0.$$

Now (12) is in turn equivalent to $w^t b \geq L - G(w)$ for all $w \geq 0$, i.e.,

(13) $$G(w) - w^t(-b) \geq L \quad \text{for all } w \geq 0.$$

Finally, (13) is equivalent to $-G^c(-b) \geq L$, where G^c is the conjugate function for G [7].

We now give some corollaries of the previous results, where the cases (a) and (b) referred to are those of the introduction.

Corollary 1: There exist $\lambda \geq 0$ such that $v(D_\lambda) \geq L$ and (a) is satisfied iff $L \leq \inf_{\epsilon \geq 0} \{M\epsilon + h(\epsilon)\}$, where $h(\epsilon) = \inf \{f(x) \mid g_i(x) \leq \epsilon \text{ for } 1 \leq i \leq k \text{ and } x \in S\}$.

Proof: We take A to be the $1 \times k$ matrix of ones and apply Theorem 1. For any $u \in R^k$ let $\alpha = \max \{0, \max \{u_i \mid 1 \leq i \leq k\}\}$. Then $V(u) = M\alpha$ and $p(u) \geq p(\alpha,\ldots,\alpha)$ so $\inf_u \{V(u) + p(u)\} = \inf_{\epsilon \geq 0} \{M\epsilon + h(\epsilon)\}$.

Q.E.D.

We note that Corollary 1 also follows directly from Theorem 2, since, for a $1 \times k$ matrix of ones, $G = h$ where G is defined in (10) and h is defined in Corollary 1. Then the condition $w^t b \geq L - G(w)$ for all $w \geq 0$, of the discussion following Theorem 2, becomes $w_1 M \geq L - h(w_1)$ for all $w_1 = \epsilon \geq 0$, and we obtain Corollary 1.

Corollary 2: There exist λ such that $v(D_\lambda) \geq L$ and (b) is satisfied iff $L \leq \inf_{u \geq 0} \{M(\Sigma u_i) + p(u)\}$.

Proof: Here A is an identity matrix and $V(u) = M \sum_{i=1}^{k} \max \{u_i, 0\}$. If $u'_i = \max \{u_i, 0\}$, $V(u') = V(u)$ and $p(u') \leq p(u)$, so we need only look at those $u \geq 0$ in taking $\inf_u \{V(u) + p(u)\}$.

Q.E.D.

Corollary 2 can also be obtained directly from Theorem 2; we omit the proof.

To give an application of these results in a computational setting, we take Theorem 2 for the case of one row $(m = 1)$; in particular, we focus on the case (a) of the introduction for $M = 1$ (though the case of a constraint $\sum_1^k a_i \lambda_i \leq 1$, for all $a_i > 0$, is similarly treated). Since a perturbation function $h(\epsilon)$ of Corollary 1 is involved in obtaining

upper bounds on $v(D_\lambda)$, which already embodies a significant amount of information on the primal problem (P) (note that $h(0)$ alone is the value of (P)), our bounds will necessarily require some knowledge of the primal problem. It turns out, however, that to get bounds on $h_b(A,L)$ of Theorem 2 in this case, does not require a knowledge of all of h, but only of $h(\epsilon_1)$ and $h(\epsilon_2)$ for values $0 < \epsilon_1 < \epsilon_2$. The analysis follows in the next paragraphs; it is straightforward.

One trivially proves that G as defined in (10) is convex on its domain. By the paragraph following Corollary 1, so is h of Corollary 1. Now assume that $h(0)$ is defined (i.e., (P) has finite value $v(P)$) and $h(\epsilon)$ is defined for at least one $\epsilon > 0$. (From Theorem 2, if $h(\epsilon) = -\infty$ for any $\epsilon > 0$, then $v(D_\lambda) = -\infty$ for all $\lambda \geq 0$.) By convexity, $h(\epsilon)$ is then defined for all $\epsilon \geq 0$, and continuous for $\epsilon > 0$; clearly, h is nonincreasing in $\epsilon \geq 0$. Therefore, the graph of h looks like the function drawn in Figure 1.

In Figure 1, we have drawn the case where $L\# = \sup\{h(\epsilon) \mid \epsilon > 0\} < h(0) = v(P)$, which can occur if h is discontinuous at its boundary point $\epsilon = 0$ (of course, $L\# = h(0) = v(P)$ is forced if there is a Slater point for (P), for then $h(\epsilon)$ is defined for some $\epsilon < 0$ and 0 is an interior point of the domain of h, hence a point of continuity). This need not be the case; if h is continuous at $\epsilon = 0$, $L\# = h(0) = v(P)$. Also in Figure 1, we have $h'(0) = -\infty$, where $h'(0)$ indicates the right directional derivative of h at 0, which exists as h is convex. This also need not be the case, for $h'(0) > -\infty$ is possible, and then (as $h'(0) \leq 0$ by monotonicity) $h'(0)$ would be finite. Thus there are actually four possibilities at $\epsilon = 0$ in drawing the graph of h

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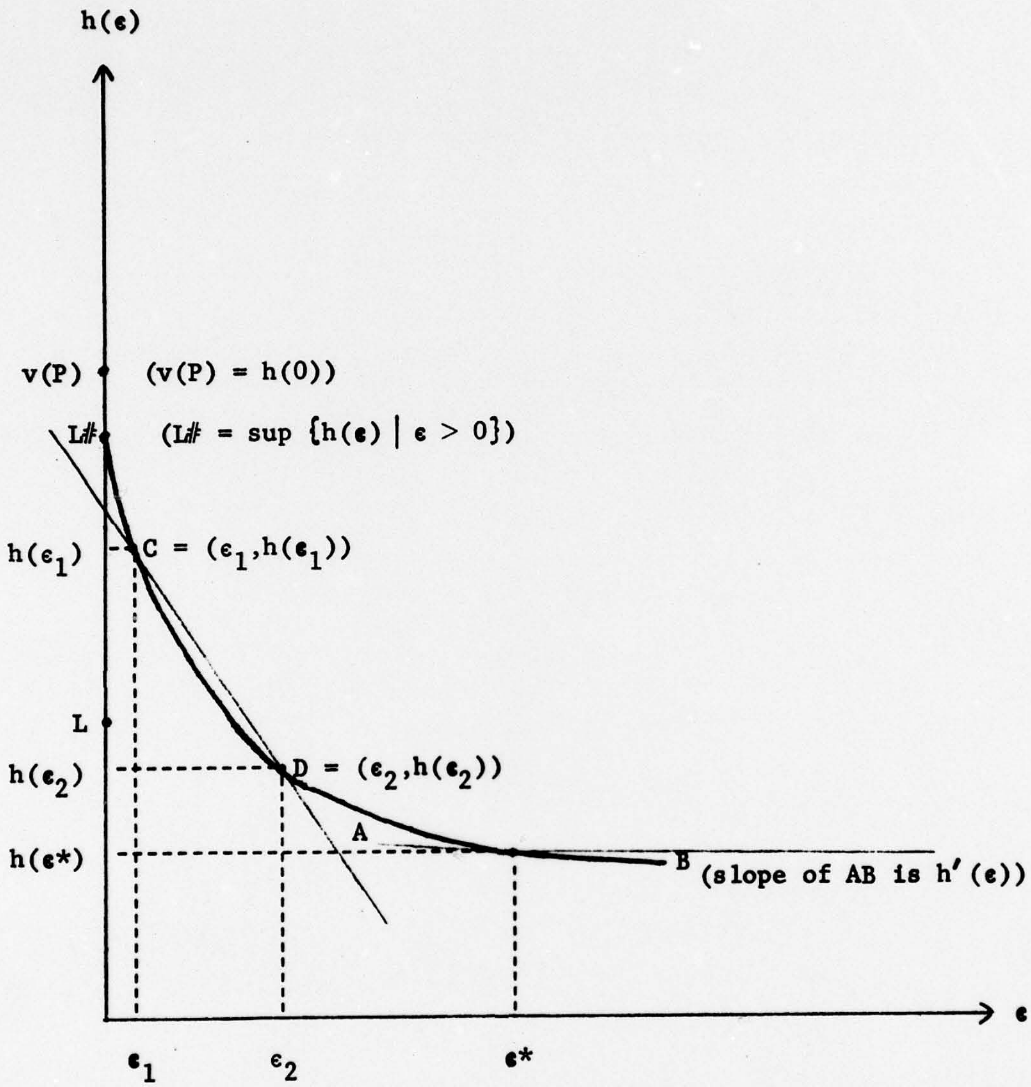
h(ε)
v(P)
(v(P) = h(0))
L#
(L# = sup {h(ε) | ε > 0})
h(ε1)
C = (ε1,h(ε1))
L
h(ε2)
D = (ε2,h(ε2))
A
h(ε*)
B
(slope of AB is h′(ε))
ε
ε1
ε2
ε*

Figure 1

(corresponding to $L\# < v(P)$ or $L\# = v(P)$, and $h'(0) = -\infty$ or $h'(0) > -\infty$), of which we have drawn only one. Furthermore, in our picture h has a horizontal asymptote, which simply means that the criterion function f of (P) is bounded below; but this need not be the case in general.

Note that P_α, as defined above Theorem 2, is always nonempty for all $\alpha \geq 0$, for the constraint (a) with $M = 1$. By Theorem 2, there exists λ with $\lambda \geq 0$ and $\sum_1^k \lambda_i \leq \alpha$ if and only if $\alpha \geq h_b(A,L)$. The supremum quotient on the r.h.s. in (11) has an interesting interpretation; here $w = (w_1)$ and $b = (b_1) = (1)$.

For this quotient to be finite, one certainly requires $(L - G(0))/(0 \cdot 1) = (L - h(0))/0 < +\infty$, which, by the conventions in Theorem 2, necessitates $L \leq h(0) = v(P)$. This fact is well known: the dual value $v(D)$ cannot exceed $v(P)$.

Moreover, in addition to $L \leq v(P)$, for this quotient to be finite for $L = L\#$, requires that the right-sided directional derivative $\dot{g}'(\epsilon)$ of the function $g(\epsilon)$ be finite at zero, where g is the continuous convex function defined by:

$$g(\epsilon) = \begin{cases} L\#, & \text{if } \epsilon = 0; \\ h(\epsilon), & \text{if } \epsilon > 0. \end{cases} \tag{14}$$

Indeed, the supremum quotient on the r.h.s. in (11), for $w = (w_1) > 0$, is simply the negative of this right-sided directional derivative. Thus, when (as in Figure 1) this directional derivative is $-\infty$, one has $v(D_\lambda) < L\#$ for all $\lambda \geq 0$ (as $\alpha = +\infty$ is forced, by Theorem 2); when this directional derivative is finite, there is $\lambda \geq 0$ with $v(D_\lambda) = L\#$ and in fact $\sum_1^k \lambda_i \leq -g'(0) = h_1(A,L\#)$ (by Theorem 2). By the same reasoning,

one cannot have $v(D_\lambda) \geq L$ for $\lambda \geq 0$, if $L > L$#, for then the r.h.s. in (11) must be unbounded: this is a consequence of the well-known fact that $\lim \{p(\epsilon_1,\ldots,\epsilon_k) \mid \epsilon_1,\ldots,\epsilon_k \searrow 0^+\}$ is the value of the dual in (P).

Now the analysis of the previous paragraph extends to a point $\epsilon^* > 0$, for this yields a convex program when translated to the origin (using convex constraints $g_i(x) - \epsilon^* \leq 0$, for $1 \leq i \leq k$). Thus, the negative of the right-sided directional derivative of g at $\epsilon = \epsilon^*$ (which is that of h, by continuity at $\epsilon = \epsilon^*$) is a lower bound on $\sum_1^k \lambda_i$ for the translated program. By convexity of h, this directional derivative is not less than the directional derivative at $\epsilon = 0$. Hence, a lower bound on $\sum_1^k \lambda_i$ for (P) is $-h(\epsilon^*)$ for any $\epsilon^* > 0$. (Since $h(\epsilon)$ is defined also for $0 < \epsilon < \epsilon^*$, the same is true of the left-sided directional derivative, by convexity of h, and the left-sided derivative of course gives a better lower bound. In Figure 1, we have drawn the case that h is differentiable at $\epsilon = \epsilon^*$.)

Now the directional derivative of a perturbation function can be hard to compute, but again the convexity of h simplifies the matter. In fact, if we know $h(\epsilon_1)$ and $h(\epsilon_2)$ for two values $0 < \epsilon_1 < \epsilon_2$, then a lower bound on $\sum_1^k \lambda_i$ is given by the negative of the slope of the secant line CD, to whit,

(15) $$(h(\epsilon_1) - h(\epsilon_2))/(\epsilon_2 - \epsilon_1).$$

Indeed, this secant line has slope not less than $h'(\epsilon_1)$, and $-h'(\epsilon_1)$ is a lower bound on $\sum_1^k \lambda_i$. Clearly, the lower bound given by (15) improves as ϵ_1 nears zero, and $\epsilon_2 > \epsilon_1$ is nearer to ϵ_1.

However, all the quantities (15) are lower bounds on $\sum_1^k \lambda_i$. If, in a computational setting, any of these bounds (15) is excessively large,

one knows that the theoretically optimal dual value $L\# = v(0)$ cannot be computationally attained, and one will have to settle for a lower value $L < L\#$.

For $L < L\#$, the supremum quotient in equation (11) clearly gives the negative of the slope of a line through $(0,L)$ that is tangent to the graph of $h(\cdot)$, or, lacking such a point of tangency, asymptotic to it or parallel to an asymptote of it. Clearly, if the directional derivative at $\epsilon = \epsilon_2 > 0$ intersects the vertical axis in Figure 1 at a point not below L, then ϵ_2 is at or to the left of this point of tangency, if it exists (if an asymptote exists, we view all $\epsilon > 0$ as "to the left of" the "point of tangency"). Hence an <u>upper</u> bound on $\sum_1^k \lambda_i$ to attain $v(D_\lambda) \geq L$ is $-h'(\epsilon_2)$, so another (generally weaker) upper bound is given by (15). Thus, if the value of (15) is computationally feasible, it yields a number L such that $v(D_\lambda) \geq L$ for some $\lambda \geq 0$ with $\sum_1^k \lambda_i$ computationally feasible.

Section 2: $\{\lambda \mid A\lambda \leq b,\ \lambda \geq 0\}$ may be unbounded

One way of approaching the case treated in this section is by a simple reduction to the bounded case of the previous section. Again $P = \{\lambda \mid A\lambda \leq b,\ \lambda \geq 0\}$, $V(u)$ is as defined in equation (3), and $p(u)$ is defined as in equation (4). We shall also need the notations:

$$(16) \qquad P'_M = \{\lambda \mid A\lambda \leq b,\ \lambda \geq 0,\ \sum_1^k \lambda_i \leq M\}$$

and

$$(17) \qquad V'_M(u) = \max\ \{u\lambda \mid \lambda \in P'_M\}.$$

Here is a result one can obtain as an immediate application of Theorem 1.

<u>Corollary 3</u>: Suppose that $P \neq \emptyset$, and set

$$L^* = \lim_{M\to+\infty} \inf_{u\in R^k} \{V'_M(u) + p(u)\}. \tag{18}$$

If $L^* \in R \cup \{-\infty\}$, then $v(D_\lambda) \leq L^*$ for all λ with $\lambda \geq 0$ and $A\lambda \leq b$.

Moreover, if $L^* \in R$ and $L' < L^*$, there exists $\underline{\lambda} \geq 0$ with $A\underline{\lambda} \leq b$ and $v(D_{\underline{\lambda}}) \geq L'$.

Finally, $L^* \in R \cup \{-\infty\}$.

<u>Proof</u>: Clearly, $V'_M(u)$ is monotone nondecreasing in M for any $u \in R^k$, hence the same is true of $\inf_{u\in R^k} \{V'_M(u) + p(u)\}$. Thus, the limit indicated in (18) exists.

If $L^* = -\infty$, then by monotonicity $\inf_{u\in R^k} \{V'_M(u) + p(u)\} = -\infty$ for all $M \geq 0$. If $\lambda \geq 0$ *satisfies* $A\lambda \leq b$, *then* $\lambda \in P'_M$ for sufficiently large M, and Theorem 1 applies, showing $v(D_\lambda) = -\infty$.

If $L^* \in R$, let $L\# > L^*$. By monotonicity, $\inf_{u\in R^k} \{V'_M(u) + p(u)\} < L\#$ for all M. If $\lambda \geq 0$ satisfies $A\lambda \leq b$, then $\lambda \in P'_M$ for sufficiently large M, and by Theorem 1 we have $v(P_\lambda) \leq L\#$. Since $v(P_\lambda) \leq L\#$ holds for all $L\# > L^*$, actually we have $v(P_\lambda) \leq L^*$ also.

Next, if $L' < L^*$, by monotonicity, for sufficiently large M, $\inf_{u\in R^k} \{V'_M(u) + p(u)\} > L'$. By Theorem 1, for M_0 sufficiently large, there is $\underline{\lambda} \in P'_{M_0} \subseteq P$ with $v(D_{\underline{\lambda}}) = \inf_{u\in R^k} \{V'_{M_0}(u) + p(u)\} > L'$.

Finally, by the last paragraph, if $L^* = +\infty$ then $v(D_{\underline{\lambda}}) > v(P)$ for some $\underline{\lambda} \in P$, since (P) is assumed consistent. However, as $v(D) \leq v(P)$, and $v(D) = \sup \{v(D_\lambda) \mid \lambda \geq 0\}$, we have a contradiction.

Q.E.D.

The proof of Corollary 3 makes it evident, that there is $\underline{\lambda} \in P$ with $v(D_{\underline{\lambda}}) = L^*$ if and only if the limiting operation in (18) is vacuous, i.e., if and only if

$$L^* = \inf_{u \in R^k} \{V'_M(u) + p(u)\} \tag{19}$$

for all sufficiently large M.

It is clear that

$$V'_M(u) + p(u) \leq V(u) + p(u) \tag{20}$$

for all $u \in R^k$, where one may interpret $\infty - \infty$ on the r.h.s. of (20) to be $-\infty$. Taking the infimum over $u \in R^k$ and then applying the limiting operation to both sides of (20) yields

$$L^* \leq \inf_{u \in R^k} \{V(u) + p(u)\}. \tag{21}$$

However, unlike the bounded case, strict inequality can occur in (21), i.e., one can have

$$\sup \{v(D_\lambda) \mid \lambda \in P\} < \inf_{u \in R^k} \{V(u) + p(u)\} \tag{22}$$

as our next example shows.

<u>Example</u>. Let $P = \{\lambda \mid \lambda \geq 0\}$. Then we have for $u \in R^k$, $u = (u_1, u_2)$,

$$V(u) = \begin{cases} 0, & \text{if } u_1 \leq 0 \text{ and } u_2 \leq 0; \\ +\infty, & \text{if } u_1 > 0 \text{ or } u_2 > 0; \end{cases} \tag{23}$$

and

$$V'_M(u) = \begin{cases} 0, & \text{if } u_1 \leq 0 \text{ and } u_2 \leq 0; \\ M \max \{u_1, u_2\}, & \text{if } u_1 > 0 \text{ or } u_2 > 0. \end{cases} \tag{24}$$

We calculate, using the fact that p(u) is monotone non-increasing in u,

$$(25)\qquad \inf_{u\in R^k} \{V(u) + p(u)\} = \inf \{p(u) \mid u_1 \leq 0 \text{ and } u_2 \leq 0\} = p(0)$$

provided that p(u) is never $-\infty$.

Now p(0) is the value v(P) of the primal problem (P), while $v(D) = \sup \{v(D_\lambda) \mid \lambda \geq 0\}$ is the value of the dual. As is well-known, one can have $v(D_\lambda) < v(P)$ even if p(u) is never $-\infty$.

For example, in R^2 with $S = \{(w_1,w_2) \mid w_1 \geq 0 \text{ and } w_2 \geq 0\}$, $f(x) = f(w_1,w_2) = \max \{-1,-\sqrt{w_1w_2}\}$ (recall that $\sqrt{w_1w_2}$ is concave), k = 1, $g_1(x) = g_1(w_1,w_2) = w_1$, we have v(P) = p(0) = 0. However, $p(u) = \inf \{\max \{-1,-\sqrt{w_1w_2}\} \mid w_1 \leq u\}$ is such that $p(u) = +\infty$ if $u < 0$, $p(u) = -1$ if $u > 0$ (since $-\sqrt{uw_2} \leq -1$ for $w_2 > 0$ sufficiently large, whenever $u > 0$). Therefore $v(D) = \lim \{p(u) \mid u \downarrow 0^+\} = -1 < v(P)$ (or one may demonstrate that v(D) = -1 directly). In this example, all functions are continuous on S.

Without introducing some form of "constraint qualification" which rules out examples of the type just given, there is another result, beyond that of Corollary 3, which involves a different analysis. First we need a preliminary lemma on semi-infinite systems of linear inequalities.

<u>Lemma 2</u>: Let $a^i \in R^n$, $b_i \in R$ for $i \in I$, $I \neq \emptyset$. If the semi-infinite system

$$(26)\qquad a^i x \geq b\ , \quad i \in I$$

is inconsistent, and $F \subseteq I$ is a finite subset of I, there exists $c \in R^n$, $d \in R$ such that $cx \geq d$ is obtained as a nonnegative linear combination of the inequalities $a^i x \geq b_i$, $i \in F$ (plus possibly a decrease in the right-hand-side), with the property that the semi-infinite system

$$(26)' \qquad \begin{aligned} a^i x &\geq b_i, \quad i \in I \setminus F \\ cx &\geq d \end{aligned}$$

is also inconsistent.

In particular, for F fixed and finite, $F \subseteq I$, if every system of the form $(26)'$ is consistent, with $cx \geq d$ obtained as described, then (26) is also consistent.

<u>Proof</u>: Let $Q_1 = \{x \mid a^i x \geq b_i, i \in F\}$ and let $Q_2 = \{x \mid a^i x \geq b_i, i \in I \setminus F\}$. Both Q_1 and Q_2 are closed convex sets, and $Q_1 \cap Q_2 = \emptyset$ by the inconsistency of (26).

Suppose first that $Q_1 \neq \emptyset$. Let $cx \geq d$ be a hyperplane which strictly separates Q_1 from Q_2, i.e., $cx \geq d$ for $x \in Q_1$ and $cx < d$ for $x \in Q_2$. Then since $a^i x \geq b_i$, $i \in F$, implies $cx \geq d$ and $Q \neq \emptyset$, $cx \geq d$ is a nonnegative combination of the inequalities of Q_1, with possibly a decrease in the r.h.s. d. Since $cx < d$ for all $x \in Q_2$, $(26)'$ is inconsistent.

Next, suppose $Q_1 = \emptyset$. Then $0 \cdot x \geq 1$ is obtainable as a nonnegative combination of the constraints of Q_1. Take $c = 0$, $d = 1$ in $(26)'$, and note that $(26)'$ is clearly inconsistent.

Q.E.D.

The following is a version of the alternative result that we mentioned above Lemma 2; we will rephrase it in final form as Theorem 4 below (compare Lemma 3 with Lemma 1).

<u>Lemma 3</u>: Suppose that $P \neq \emptyset$.

There is no $\lambda \in P$ with $v(D_\lambda) \geq L$ if and only if there is a $u \in R^k$ with $V(u) < +\infty$ such that for every M, there is an $\underline{x}$ such that the finite system of inequalities

$$(27)_{\underline{x},M,u} \qquad u\lambda \leq V(u)$$

$$\sum_{i=1}^{k} g_i(\underline{x})\lambda_i \geq L - f(\underline{x})$$

$$\sum_{i=1}^{k} \lambda_i \leq M$$

$$\lambda \geq 0$$

is inconsistent.

__Proof__: The existence of $\lambda \in P$ such that $v(D_\lambda) \geq L$ is precisely equivalent to the consistency of the semi-infinite system

$$(28) \qquad A\lambda \leq b$$

$$\sum_{i=1}^{k} g_i(x)\lambda_i \geq L - f(x), \ x \in S$$

$$\lambda \geq 0.$$

By Lemma 2, (28) is inconsistent if and only if there exists a vector of multipliers $\theta \geq 0$ such that

$$\theta A\lambda \leq \theta b$$

$$(28)' \qquad \sum_{i=1}^{k} g_i(x)\lambda_i \geq L - f(x), \ x \in S$$

$$\lambda \geq 0$$

is inconsistent. Put $u = \theta A$; since $V(u) \leq \theta b$, we see that $V(u) < +\infty$, and $(28)'$ remains inconsistent with $V(u)$ replacing θb.

Thus, for any M, the following semi-infinite system is inconsistent:

$$(28)'' \qquad u\lambda \leq V(u)$$

$$\sum_{i=1}^{k} g_i(x)\lambda_i \geq L - f(x),\ x \in S$$

$$\sum_{i=1}^{k} \lambda_i \leq M$$

$$\lambda \geq 0.$$

Since the polyhedron $Q_M = \{\lambda \geq 0 \mid \sum_{i=1}^{k} \lambda_i \leq M,\ u\lambda \leq V(u)\}$ is either empty or nonempty and bounded, by Lemma 1 (with P replaced by Q_M) there is an $\underline{x}$ such that $(27)_{\underline{x},M,u}$ is inconsistent.

Q.E.D.

As preliminary remarks to Theorem 4 just below, we note that always $V(u) > -\infty$ since $P \neq \emptyset$. Also, $V(u)$ is convex, as is well-known (for example, note that $V(u) = \min \{\theta b \mid \theta A \geq u,\ \theta \geq 0\}$ if $V(u) < +\infty$, from duality; and a minimizing linear program is convex in its r.h.s. u). Clearly, $V(\alpha u) = \alpha V(u)$ for all $\alpha \geq 0$, when $V(u) < +\infty$.

Lemma 3 can be rephrased, in terms of the perturbation function p, as follows.

<u>Theorem 4</u>: Suppose that $P \neq \emptyset$. Define

$$(29) \qquad J = \inf_{\substack{u \in R^k \\ V(u) < +\infty}} \ \lim_{\beta \to 0^+} \ \inf_{\alpha \geq 0} \{\alpha V(u) + p(\alpha u + \beta e)\}$$

where $e = (1,\ldots,1)^T$ is a vector of one's. Then:

(30a) If $L < J$, there exists $\lambda \in P$ with $v(D_\lambda) \geq L$.

(30b) If $L > J$, there does not exist $\lambda \in P$ with $v(D_\lambda) \geq L$.

i.e., $J = L^*$, where L^* is given in equation (18).

<u>Proof</u>: We first prove (30a). Fix $u \in R^k$ with $V(u) < +\infty$. Define the function

$$h_u(\beta) = \inf_{\alpha \geq 0} \{\alpha V(u) + p(\alpha u + \beta e)\}. \tag{31}$$

We note that h_u is a convex function of β. In fact, let $\beta_1, \beta_2 > 0$ be given and let $p_1 > h_u(\beta_1)$, $p_2 > h_u(\beta_2)$. Then there exists $\alpha_1, \alpha_2 \geq 0$ with

$$p_1 > \alpha_1 V(u) + p(\alpha_1 u + \beta_1 e) \tag{32a}$$

$$p_2 > \alpha_2 V(u) + p(\alpha_2 u + \beta_2 e). \tag{32b}$$

Now if $0 \leq \lambda \leq 1$, using the convexity of V and p, we obtain from (32)

$$\begin{aligned} \lambda p_1 + (1-\lambda)p_2 &> \lambda V(\alpha_1 u) + (1-\lambda)V(\alpha_2 u) \\ &\quad + \lambda p(\alpha_1 u + \beta_1 e) + (1-\lambda)p(\alpha_2 u + \beta_2 e) \\ &\geq V((\lambda\alpha_1 + (1-\lambda)\alpha_2)u) \\ &\quad + p((\lambda\alpha_1 + (1-\lambda)\alpha_2)u + (\lambda\beta_1 + (1-\lambda)\beta_2)e) \\ &\geq \inf_{\alpha \geq 0} \{V(\alpha u) + p(\alpha u + (\lambda\beta_1 + (1-\lambda)\beta_2)e)\} \\ &= h_u(\lambda\beta_1 + (1-\lambda)\beta_2). \end{aligned} \tag{33}$$

Since $p_1 > h_u(\beta_1)$ and $p_2 > h_u(\beta_2)$ were arbitrary, from (33) we obtain $\lambda h_u(\beta_1) + (1-\lambda)h_u(\beta_2) \geq h_u(\lambda\beta_1 + (1-\lambda)\beta_2)$ and our proof of the convexity of $h_u(\cdot)$ is complete.

Let $L < J$. Then, by the definition of J as the r.h.s. of (29), also $L < \lim_{\beta \searrow 0^+} h_u(\beta)$. Using the convexity of $h_u(\cdot)$, $h_u(\beta) \geq L - M\beta$ for all $\beta \geq 0$, provided only that M is chosen sufficiently large (M need only not be less than the negative of the slope of the line through (0,L), which

is either tangent or asymptotic to the curve given by $v = h_u(\beta)$ for $\beta \geq 0$). Since $P \neq \emptyset$, we can also assume that there is $\lambda \in P$ with $\sum_1^k \lambda_i \leq M$.

We now prove that, for any $\underline{x} \in S$, the system $(27)_{\underline{x},M,u}$ is consistent. Then by Lemma 3, there is $\lambda \in P$ with $v(D_\lambda) \geq L$ and we have (30a).

Now if $(27)_{\underline{x},M,u}$ is inconsistent, let $\alpha \geq 0$ be the multiplier on the first row, $\delta \geq 0$ the multiplier for the second row, and $\beta \geq 0$ the multiplier on the third row, such that, by the Kuhn-Fourier Theorem [8],

(34a) $$\alpha u - \delta(g_1(\underline{x}),\ldots,g_k(\underline{x})) + \beta e \geq 0$$

(34b) $$\alpha V(u) + \delta(f(\underline{x}) - L) + \beta M < 0.$$

In (34), we must have $\delta > 0$, since the system $(27)_{\underline{x},M,u}$, with the second row deleted, is consistent (let $x \in P$ be such that $\sum_1^k \lambda_i \leq M$; then λ solves this system). Without loss of generality, $\delta = 1$.

Now (34a) becomes

(35) $$(g_1(\underline{x}),\ldots,g_k(\underline{x})) \leq \alpha u + \beta e$$

so that $f(\underline{x}) \geq p(\alpha u + \beta e)$. Combining this with (34b), we obtain

(36) $$\alpha V(u) + p(\alpha u + \beta e) < L - \beta M$$

from which follows $h_u(\beta) < L - \beta M$. However, from the choice of M, $h_u(\beta) \geq L - \beta M$, and we have a contradiction. Thus $(27)_{\underline{x},M,u}$ cannot be inconsistent.

We next prove (30b). First assume $J > -\infty$. Let $L > J$, and choose n so that $L \geq J + 3/n$. Then for some $u \in R^k$ with $V(u) < +\infty$, we have $\lim_{\beta \searrow 0^+} h_u(\beta) \leq J + 1/n$. Let M be arbitrary. For all β sufficiently small, there exists an $\alpha \geq 0$ with

(36) $$\alpha V(u) + p(\alpha u + \beta e) \leq J + 2/n.$$

By choosing β so small that $\beta M < 1/n$, we have

(37) $$\alpha V(u) + p(\alpha u + \beta e) + \beta M < L.$$

From (37), there exists a vector $\underline{x} \in S$ with

(38a) $$(g_1(\underline{x}),\ldots,g_k(\underline{x})) \leq \alpha u + \beta e$$

(38b) $$\alpha V(u) + (f(\underline{x}) - L) + \beta M < 0.$$

Now (38) shows that $(27)_{\underline{x},M,u}$ is inconsistent. Since M was arbitrary, Lemma 3 shows that $v(D_\lambda) \geq L$ is impossible for $\lambda \in P$.

In the case $J = -\infty$, (37) is easily obtained, and the same argument will work.

Q.E.D.

Our next and final characterization of $\sup\{v(D_\lambda) \mid \lambda \in P\}$ for the case of $P \neq \emptyset$ and P possibly unbounded, is a direct application of Theorem 3. First we define the altered perturbation function, depending on A, by:

(39) $$H(w,w_{m+1}) = \inf\{f(x) \mid \text{subject to } x \in S \text{ and } (g_1(x),\ldots,g_k(x)) \leq w_tA + w_{m+1}e\}$$

where $(w,w_{m+1}) \in R^{m+1}$, $w \in R^m$, and $e = (1,\ldots,1)^T$ is again the vector of one's (compare (39) with (10)). Now (39) is easily seen to be the previous perturbation function of type (10) for the constraint system

(40) $$\begin{pmatrix} A\lambda \\ \sum_1^k \lambda_i \end{pmatrix} \leq \begin{pmatrix} b \\ \gamma \end{pmatrix} + \alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in which a row $\sum_1^k \lambda_i \leq 1$ has been added to the constraints of P to obtain a bounded polyhedron, and $\gamma \in R$.

Theorem 5: Suppose that $P \neq \emptyset$ and $b \geq 0$.

There exists $\lambda \in P$ such that $v(D_\lambda) \geq L$ if and only if the supremum

$$\sigma = \sup_{w, w_{n+1} \geq 0} (L - H(w, w_{n+1}) - wb)/w_{n+1} \tag{41}$$

is finite. When this supremum is finite, in fact there is $\lambda \in P$ with $v(D_\lambda) \geq L$ and $\sum_1^k \lambda_i = \sigma$; and there is no $\lambda \in P$ with $v(D_\lambda) \geq L$ and $\sum_1^k \lambda_i < \sigma$.

Proof: We apply Theorem 3 with $\binom{b}{\gamma}$ replacing b, and with $\binom{0}{1}$ replacing c, where $\gamma = \min \{\sum_1^k \lambda_i \mid \lambda \in P\}$.

Q.E.D.

It is worth noting that, in Theorems 4 and 5 (the latter with H as defined in (39)), the vector e can be replaced by any vector with all components strictly positive. In fact, the same proofs are valid with this change.

June 16, 1978

References

1. C. E. Blair, "Convex Optimization and Lagrange Multipliers," to appear in Mathematical Programming,

2. A. Charnes, W. W. Cooper, and K. O. Kortanek, "On Representations of Semi-Infinite Programs Which Have No Duality Gaps," Management Science, vol. 12, no. 1, September 1965, pp. 113-121.

3. A. Charnes, W. W. Cooper, and K. O. Kortanek, "Duality, Haar Programs, and Finite Sequence Spaces," Proceedings of the National Academy of Sciences, U.S.A., vol. 68 (1962), pp. 605-608. (See also [2].)

4. R. J. Duffin, "Convex Analysis Treated by Linear Programming," Mathematical Programming, vol. 4, no. 2, 1973, pp. 125-143.

5. R. J. Duffin and R. G. Jeroslow, "Affine Minorants and Lagrangean Functions," in preparation.

6. R. J. Duffin and L. A. Karlovitz, "An Infinite Linear Program with a Duality Gap," Management Science, vol. 12, no. 1, September 1965, pp. 122-134.

7. R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, New Jersey, 1970, 432+ pp.

8. J. Stoer and C. Witzgall, Convexity and Optimization in Finite Dimensions I, Springer-Verlag, New York, 1970, 268+ pp.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER M.S.R.R. 419	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Lagrange Dual Problems with Linear Constraints on the Multipliers		5. TYPE OF REPORT & PERIOD COVERED June, 1978
		6. PERFORMING ORG. REPORT NUMBER M.S.R.R. 419
7. AUTHOR(s) C.E. Blair and R.G. Jeroslow		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0621
9. PERFORMING ORGANIZATION NAME AND ADDRESS GSIA Carnegie-Mellon University, Pittsburgh, PA		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 047-048
11. CONTROLLING OFFICE NAME AND ADDRESS Personnel and Training Research Programs Office of Naval Research (Code 458) Arlington, Virginia 22217		12. REPORT DATE June, 1978
		13. NUMBER OF PAGES 26
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Lagrangeans, Duality, Convexity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We consider Lagrange dual problems in which the vector of Lagrange multipliers, in addition to the usual nonnegativity conditions, satisfies certain linear constraints. Of course, the maximum dual value, for such restricted classes of multipliers, need not equal the primal value, even if various "constraint qualifications" are appended.

We obtain expressions for this maximum "constrained dual" value, in terms of perturbation functions for the primal convex program, and value functions

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associated with the linear constraints. At least some information on the primal perturbation functions appears to be necessary in order to put upper or lower bounds on the "constrained dual" value. For the case that the constraints are, simply, that a positive weighted sum of the multipliers shall not exceed some given bound (plus, of course, the usual nonnegativities), only two values of an associated one-dimensional perturbation function is needed to obtain such upper and lower bounds.

Unclassified

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